

AN EXAMPLE CONCERNING HAMILTONIAN GROUPS OF SELF PRODUCT, II

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ABSTRACT. We describe the natural identification of $FH_*(X \times X, \Delta; \omega \oplus -\omega)$ with $FH_*(X, \omega)$. Under this identification, we show that the extra elements in $\text{Ham}(X \times X, \omega \oplus -\omega)$ found in [3], for $X = (S^2 \times S^2, \omega_0 \oplus \lambda\omega_0)$ for $\lambda > 1$, do not define new invertible elements in $FH_*(X, \omega)$.

AMS Subject Classification: 53D12; 53D40, 57S05

Keywords: Lagrangian submanifolds, Hamiltonian group, Seidel elements.

1. INTRODUCTION

Let M be a symplectic manifold with an anti-symplectic involution c , such that L is the Lagrangian submanifold fixed by c . For any map $u : (\Sigma, \partial\Sigma) \rightarrow (M, \Delta)$, where Σ is a manifold with boundary, we define $v : \Sigma \cup_{\partial} \bar{\Sigma} \rightarrow X$ by

$$v|_{\Sigma} = p_1 \circ u \text{ and } v|_{\bar{\Sigma}} = p_2 \circ u,$$

where $\bar{\Sigma}$ is Σ with the opposite orientation. For any map $v : \Sigma \cup_{\partial} \bar{\Sigma} \rightarrow X$ we obtain the corresponding map $u : (\Sigma, \partial\Sigma) \rightarrow (M, \Delta)$ by

$$u(x) = (v(x), v(\bar{x})),$$

where \bar{x} denotes $x \in \bar{\Sigma}$. We use δ to denote the map $v \mapsto u$.

For $M = X \times X$ and involution switching the factors, then $L = \Delta$. Let $\delta_k := p_k \circ \delta$ where p_k is the projection to the k -factor, then it induces a map of Floer homologies. We show in §2

Lemma 1.1. *There is a commutative diagram of isomorphisms of the Floer homologies:*

$$\begin{array}{ccccc} & & FH_*(M, \Delta) & & \\ & \delta_1 & \simeq & \delta_2 & \\ & \nwarrow & & \nearrow & \\ FH_*(X) & \xleftarrow{\quad} & & \xrightarrow{\quad} & FH_*(\bar{X}) \\ & \tau & & & \\ & \simeq & & & \end{array}$$

In [2], with proper assumptions, we described a construction of Lagrangian Seidel element from a path of Hamiltonian diffeomorphisms. In particular, a loop γ in $\text{Ham}(M)$ defines a Lagrangian Seidel element $\Psi_{\gamma}^L \in FH_*(M, L)$, where L is a Lagrangian submanifold. The Albers' map

$$\mathcal{A} : FH_*(M) \rightarrow FH_*(M, L)$$

whenever well-defined, for example when L is monotone, relates the Seidel elements $\Psi_{\gamma}^M \in FH_*(M)$ to Ψ_{γ}^L . Let S_M denote the image of Seidel map $\Psi^M : \pi_1 \text{Ham}(M) \rightarrow FH_*(M)$ and S_L that of $\Psi^L : \pi_1(\text{Ham}(M), \text{Ham}_L(M)) \rightarrow FH_*(M, L)$, where $\text{Ham}_L(M)$ is the group of Hamiltonian diffeomorphisms preserving L which restrict to isotopies on L , then

$$\mathcal{A}(S_M) \subseteq S_L$$

Question 1.2. *When all the terms involved is well defined, is the inclusion $\mathcal{A}(S_M) \subseteq S_L$ (in general) proper?*

An affirmative answer to this question would imply an affirmative answer to the open question about the non-triviality of $\pi_0 \text{Ham}_L(M)$.

For the case of $L = \Delta$, since δ_1 is an isomorphism, the inclusion is equivalent to $\delta_1 \mathcal{A}(S_M) \subseteq \delta_1(S_\Delta)$ as subsets of $FH_*(X)$. In §3, we show that $S_X \subseteq \delta_1 \mathcal{A}(S_M)$. More precisely,

Theorem 1.3. [Corollary 3.2] *Let $\gamma \in \pi_1 \text{Ham}(X)$. It naturally lifts to a split element $\gamma_+ \in \pi_1 \text{Ham}(M)$, and we have $\delta_1 \mathcal{A}(\Psi_{\gamma_+}^M) = \Psi_\gamma^X$.*

As a corollary, it shows that the natural map $\pi_1 \text{Ham}(X) \times \pi_1 \text{Ham}(\overline{X}) \rightarrow \pi_1 \text{Ham}(M)$ is injective. In light of this result, we pose the following question, which is related to Question 1.2 for the special case of diagonal.

Question 1.4. *Is any inclusion in the sequence $S_X \subseteq \delta_1 \mathcal{A}(S_M) \subseteq \delta_1(S_\Delta)$ proper?*

For $X = S^2 \times S^2$ as in [3], we show in §3 that the image under $\delta_1 \mathcal{A}$ of the extra Seidel elements found in [3] is contained in S_X .

Acknowledgement. S. Hu is partially supported by an NSERC Discovery Grant.

2. IDENTIFICATION OF FLOER HOMOLOGIES

2.1. Notations. Let

$$D_+^2 = \{z \in \mathbb{C} : |z| \leq 1, \Im z \geq 0\},$$

∂_+ denote the part of boundary of D_+^2 on the unit circle, parametrized by $t \in [0, 1]$ as $e^{i\pi t}$, and ∂_0 the part on the real line, parametrized by $t \in [0, 1]$ as $2t - 1$.

Let (M, L) be a pair of symplectic manifold and a Lagrangian submanifold, and Ω is the symplectic form. For $\beta \in \pi_2(M, L)$, $\mu_L(\beta)$ denotes its Maslov number, and $\Omega(\beta)$ its symplectic area. The space of paths in M connecting points of L is

$$\mathcal{P}_L M = \{l : ([0, 1], \partial[0, 1]) \rightarrow (M, L), [l] = 0 \in \pi_1(M, L)\}$$

and the corresponding covering space with covering group $\Gamma_L = \pi_2(M, L)/(\ker \omega \cap \ker \mu_L)$ is

$$\tilde{\mathcal{P}}_L M = \{[l, w] : w : (D_+^2; \partial_+, \partial_0) \rightarrow (M; l, L)\}$$

where $(l, w) \sim (l', w') \iff l = l'$ and $\omega(w\#(-w')) = \mu_L(w\#(-w'))$. The space of contractible loops in M parametrized by \mathbb{R}/\mathbb{Z} is denoted $\Omega(M)$ and the corresponding covering space with covering group $\Gamma_\omega = \pi_2(M)/(\ker \omega \cap \ker c_1)$ is given by

$$\tilde{\Omega}(M) = \{[\gamma, v] : v : (D^2, \partial D^2) \rightarrow (M, \gamma)\}$$

where $(\gamma, v) \sim (\gamma', v') \iff \gamma = \gamma'$ and $\omega(v\#(-v')) = c_1(v\#(-v'))$. Here, ∂D^2 is parametrized as the unit circle in \mathbb{C} by $\{e^{2\pi i t} : t \in [0, 1]\}$, and $c_1 = c_1(TM)$ in some compatible almost complex structure. We denote the space of loops in M parametrized by $\mathbb{R}/T\mathbb{Z}$ and the corresponding covering space as $\Omega^{(T)}(M)$ and $\tilde{\Omega}^{(T)}(M)$ respectively, thus $\Omega(M) = \Omega^{(1)}(M)$ and $\tilde{\Omega}(M) = \tilde{\Omega}^{(1)}(M)$.

Let $H : [0, 1] \times M \rightarrow \mathbb{R}$ be a time-dependent Hamiltonian function, which defines on $\tilde{\mathcal{P}}_L M$ the action functional

$$a_H([l, w]) = - \int_{D_+^2} w^* \omega + \int_{[0, 1]} H_t(l(t)) dt,$$

where we use the convention $dH = -\iota_{X_H} \omega$ for the Hamiltonian vector fields. Similarly, a time dependent Hamiltonian function K for $t \in \mathbb{R}/T\mathbb{Z}$ defines an action functional a_K on $\tilde{\Omega}^{(T)}(M)$. We will not distinguish notations for the two types of action functionals when it is clear from the context which one is under discussion.

Given the time dependent Hamiltonian function H , let $\tilde{l} \in \tilde{\mathcal{P}}_L M$ such that l is a connecting orbit for H , then $\mu_H(\tilde{l})$ denotes the corresponding Conley-Zehnder index. Similarly, for the time

dependent Hamiltonian function K , let $\tilde{\gamma} \in \tilde{\Omega}^{(T)}(M)$ such that γ is a periodic orbit for K , then $\mu_K(\tilde{\gamma})$ denotes the corresponding Conley-Zehnder index. The following relations hold

$$\mu_H(\tilde{l}) - \mu_H(\tilde{l}') = \mu_L(w\#(-w')) \text{ and } \mu_K(\tilde{\gamma}) - \mu_K(\tilde{\gamma}') = c_1(u\#(-v'))$$

where $l = l'$ and $\gamma = \gamma'$.

2.2. Doubling construction. First we describe the doubling construction when the Lagrangian submanifold is the fixed submanifold of an anti-symplectic involution. It applies in this case since the diagonal Δ is the fixed submanifold of the involution of switching the two factors.

Let $c : M \rightarrow M$ be an anti-symplectic involution and $L \subset M$ be the fixed submanifold of τ , then it is a Lagrangian submanifold. We'll use (\mathbb{H}, \mathbb{J}) to denote a pair of 2-periodical Hamiltonian functions and compatible almost complex structures, i.e.

$$\mathbb{H} : \mathbb{R}/2\mathbb{Z} \times M \rightarrow \mathbb{R} \text{ and } \mathbb{J} = \{\mathbb{J}_t\}_{t \in \mathbb{R}/2\mathbb{Z}}.$$

Definition 2.1. *The pair (\mathbb{H}, \mathbb{J}) is c -symmetric if it satisfies*

$$\mathbb{H}_t(x) = \mathbb{H}_{2-t}(c(x)) \text{ and } \mathbb{J}_t(x) = -dc \circ \mathbb{J}_{2-t} \circ dc.$$

For such a pair, we define the halves $(H, \mathbf{J}) := (\mathbb{H}_t, \mathbb{J}_t)_{t \in [0,1]}$ and

$$(H', \mathbf{J}') := (\mathbb{H}_{1-t} \circ c, -dc \circ \mathbb{J}_{1-t} \circ dc)_{t \in [0,1]} = (\mathbb{H}_{t+1}, \mathbb{J}_{t+1})_{t \in [0,1]}.$$

The doubling map δ described in the introduction is a special case of the following construction for a symplectic manifold with an anti-symplectic involution:

Definition 2.2. *Let $u : (\Sigma, \partial\Sigma) \rightarrow (M, L)$ be a map from a manifold Σ with boundary $\partial\Sigma$, the doubled map is given by:*

$$v : \Sigma \cup_{\partial} \overline{\Sigma} \rightarrow M : v|_{\Sigma} = u \text{ and } v|_{\overline{\Sigma}} = c \circ u,$$

where $\overline{\Sigma}$ is Σ with the opposite orientation. We also write $\delta(u) := v$ which gives the doubling map between the spaces of continuous maps:

$$\delta : \text{Map}(\Sigma, \partial\Sigma; M, L) \rightarrow \text{Map}(\Sigma \cup_{\partial} \overline{\Sigma}; M).$$

In particular, we have the map between the space of paths in (M, L) and loops of period 2 in M , as well as their covering spaces:

$$\delta : \mathcal{P}_L M \rightarrow \Omega^{(2)}(M) \text{ and } \delta : \tilde{\mathcal{P}}_L M \rightarrow \tilde{\Omega}^{(2)}(M)$$

Let (\mathbb{H}, \mathbb{J}) be a c -symmetric pair and $\{\phi_t\}_{t \in [0,2]}$ the Hamiltonian isotopy generated by \mathbb{H} , then

$$(2.1) \quad \phi_t = c \circ \phi_{2-t} \circ \phi_2^{-1} \circ c \implies (c \circ \phi_2)^2 = 1.$$

Let (H, \mathbf{J}) and (H', \mathbf{J}') be the two halves of \mathbb{H} , then

$$H_t = H'_{1-t} \circ c \text{ and } J_t = -dc \circ J'_{1-t} \circ dc,$$

Let ϕ'_t denote the Hamiltonian isotopy generated by H' , then

$$\phi'_t = c \circ \phi_{1-t} \circ \phi_1^{-1} \circ c$$

It follows that if l is a Hamiltonian path generated by H connecting $x, y \in L$, then $l'(t) := c \circ l(1-t)$ is a Hamiltonian path generated by H' connecting $y, x \in L$, and the double $\gamma = \delta(l)$ is a periodic orbit for \mathbb{H} . This correspondence lifts to the covering spaces and the following holds.

Lemma 2.3. *For $\tilde{l} \in \tilde{\mathcal{P}}_L M$ let $\tilde{\gamma} = \delta(\tilde{l})$, then*

$$a_{\mathbb{H}}(\tilde{\gamma}) = 2a_H(\tilde{l}) = 2a_{H'}(\tilde{l}').$$

Moreover, if \tilde{l} is a critical point of a_H then $\tilde{\gamma}$ is a critical point of $a_{\mathbb{H}}$. If $\tilde{\gamma}$ is non-degenerate, then \tilde{l} is as well. A Floer trajectory for a_H is taken to a Floer trajectory for $a_{\mathbb{H}}$ by δ , which converges to the corresponding critical points when the trajectory has finite energy. \square

A result from [2] relates the Conley-Zehnder indices of connecting paths generated by H and H' .

Lemma 2.4 (Lemma 5.2 of [2]). *Let \tilde{l} and \tilde{l}' be respective critical points of a_H and $a_{H'}$ as above. Then $\mu_H(\tilde{l}) = \mu_{H'}(\tilde{l}')$. \square*

2.3. Index comparison. We briefly recall the definition of Conley-Zehnder index using the Maslov index of paths of Lagrangian subspaces as in Robbin-Salamon [5]. Let $\tilde{l} = [l, w]$ be a non-degenerate critical point of a_H . Then $w : D_+^2 \rightarrow M$ and $l = \partial w$ is a Hamiltonian path. There is a symplectic trivialization Φ of w^*TM given by $\Phi_z : T_{w(z)}M \rightarrow \mathbb{C}^n$ with the standard symplectic structure ω_0 on \mathbb{C}^n . Furthermore, we require that $\Phi_r(T_{w(r)}L) = \mathbb{R}^n$, for $r \in [-1, 1] \subset D_+^2$. Then the linearized Hamiltonian flow $d\phi_t$ along l defines a path of symplectic matrices

$$(2.2) \quad E_t = \Phi_{e^{i\pi t}} \circ d\phi_t \circ \Phi_1^{-1} \in Sp(\mathbb{C}^n)$$

Then the Conley-Zehnder index of \tilde{l} is given by

$$\mu_H(\tilde{l}) = \mu(E_t \mathbb{R}^n, \mathbb{R}^n)$$

where μ is the Maslov of paths of Lagrangian subspaces introduced in [5].

We continue with the notations of Lemma 2.3.

Proposition 2.5. *Suppose that all the critical points involved are non-degenerate, then*

$$(2.3) \quad \mu_H(\tilde{l}) + \mu_{H'}(\tilde{l}') - \mu_{\mathbb{H}}(\tilde{\gamma}) = \frac{1}{2} \text{sign}(Q),$$

where $Q(\bullet, *) = \Omega((1 - d\phi_2)\bullet, dc(*))$ is a quadratic form on $T_{l(0)}M$.

Proof: For notational convenience, we denote

$$\tilde{l}^+ = \tilde{l}, \tilde{l}^- = \tilde{l}', H^+ = H, H^- = H', \phi_t^+ = \phi_t \text{ and } \phi_t^- = c \circ \phi_{1-t} \circ \phi_1^{-1} \circ c \text{ for } t \in [0, 1],$$

then ϕ^\pm is the flow generated by H^\pm . Assume that we can choose the trivialization $\Phi_z : T_{v(z)}M \rightarrow \mathbb{C}^n$ of v^*TM so that $\Phi_z = c_z \circ \Phi_z \circ dc$, where $c_z : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is the complex conjugation, which takes ω_0 to $-\omega_0$. In particular, $\Phi_r(T_{v(r)}L) = c_z \circ \Phi_r \circ dc(T_{v(r)}L) = \mathbb{R}^n$ for $r \in [-1, 1]$. Define the following paths of symplectic matrices:

$$F_t = \Phi_{e^{i\pi t}} \circ d\phi_t \circ \Phi_1^{-1} \text{ for } t \in [0, 2] \text{ and } F_t^\pm = \Phi_{\pm e^{i\pi t}} \circ d\phi_t^\pm \circ \Phi_{\pm 1}^{-1} \text{ for } t \in [0, 1],$$

Then $F_t = c_z \circ F_{2-t} \circ F_2^{-1} \circ c_z$ and

$$\mu_{\mathbb{H}}(\tilde{\gamma}) = \mu((F_t, 1)\Delta, \Delta) \text{ and } \mu_{H^\pm}(\tilde{l}^\pm) = \mu(F_t^\pm \mathbb{R}^n \oplus \mathbb{R}^n, \Delta)$$

where $\Delta : \mathbb{C}^n \rightarrow \mathbb{C}^n \oplus \mathbb{C}^n$ is the diagonal and the symplectic structure on $\mathbb{C}^n \oplus \mathbb{C}^n$ is given by $\Omega_0 = \omega_0 \oplus (-\omega_0)$. We have by additivity of Maslov index:

$$\mu_{\mathbb{H}}(\tilde{\gamma}) = \mu((F_t^+, 1)\Delta, \Delta) + \mu((F_t^- \circ F_1, 1)\Delta, \Delta)$$

and the left hand side of (2.3) is the sum of the following differences:

$$\mu(F_t^+ \mathbb{R}^n \oplus \mathbb{R}^n, \Delta) - \mu((F_t^+, 1)\Delta, \Delta) \text{ and } \mu(F_t^- \mathbb{R}^n \oplus \mathbb{R}^n, \Delta) - \mu((F_t^- \circ F_1, 1)\Delta, \Delta).$$

For $F \in Sp(\mathbb{C}^n)$, $(F, 1)^{-1}\Delta = (1, F)\Delta$, thus the first difference is

$$\begin{aligned} & \mu(F_t^+ \mathbb{R}^n \oplus \mathbb{R}^n, \Delta) - \mu((F_t^+, 1)\Delta, \Delta) = \mu((1, F_t^+)\Delta, \Delta) - \mu((1, F_t^+)\Delta, \mathbb{R}^n \oplus \mathbb{R}^n) \\ & = s(\mathbb{R}^n \oplus \mathbb{R}^n, \Delta; \Delta, (1, F_1)\Delta) = s(\mathbb{R}^n \oplus \mathbb{R}^n, (1, F_1)\Delta; \Delta, (1, F_1)\Delta) \end{aligned}$$

where s is the Hörmander index (cf. [5]) and the last equality follows from the following properties of Hörmander index for Lagrangian subspaces A, B, C, D, D' :

$$(2.4) \quad \begin{aligned} s(A, B; A, C) &= s(A, B; A, C) - s(A, C; A, C) = s(C, B; A, C) \\ s(A, B; C, D) - s(A, B; C, D') &= s(A, B; D', D). \end{aligned}$$

Let $c' : \mathbb{C}^n \oplus \mathbb{C}^n \rightarrow \mathbb{C}^n \oplus \mathbb{C}^n : (z_1, z_2) \mapsto (z_2, z_1)$, then c' preserves Δ and $\mathbb{R}^n \oplus \mathbb{R}^n$ while reverses the sign of the symplectic structure, thus

$$s(\mathbb{R}^n \oplus \mathbb{R}^n, (1, F_1)\Delta; \Delta, (1, F_1)\Delta) = -s(\mathbb{R}^n \oplus \mathbb{R}^n, (F_1, 1)\Delta; \Delta, (F_1, 1)\Delta).$$

For the second difference, we get

$$\begin{aligned} & \mu(F_t^- \mathbb{R}^n \oplus \mathbb{R}^n, \Delta) - \mu((F_t^- \circ F_1, 1)\Delta, \Delta) \\ &= \mu((1, F_t^-)\Delta, (F_1, 1)\Delta) - \mu((1, F_t^-)\Delta, \mathbb{R}^n \oplus \mathbb{R}^n) \\ &= s(\mathbb{R}^n \oplus \mathbb{R}^n, (F_1, 1)\Delta; \Delta, (1, F_1^-)\Delta). \end{aligned}$$

It follows that the difference on the left side of (2.3) is

$$\begin{aligned} & s(\mathbb{R}^n \oplus \mathbb{R}^n, (F_1, 1)\Delta; \Delta, (1, F_1^-)\Delta) - s(\mathbb{R}^n \oplus \mathbb{R}^n, (F_1, 1)\Delta; \Delta, (F_1, 1)\Delta) \\ &= s(\mathbb{R}^n \oplus \mathbb{R}^n, (F_1, 1)\Delta; (F_1, 1)\Delta, (1, F_1^-)\Delta). \end{aligned}$$

We now identify the last Hörmander index as the signature. Let $L = \mathbb{R}^n \oplus \mathbb{R}^n$, $K = (F_1, 1)\Delta$ and $L' = (1, F_1^-)\Delta$, then they are pairwise transverse, by the non-degeneracy assumption. Thus in the splitting $\mathbb{C}^{2n} = L \oplus K$ we may write L' as the graph of an invertible linear map $f : K \rightarrow K^* \simeq L$ and let $\overline{KL'} = \text{graph}(tf)$, $t \in [0, 1]$ be the path of Lagrangian subspaces connecting K to L' then

$$(2.5) \quad s(L, K; K, L') = \mu(\overline{KL'}, K) - \mu(\overline{KL'}, L) = \mu(\overline{KL'}, K) = \frac{1}{2} \text{sign}(Q')$$

where $Q'(v) = \Omega_0(v, f(v))$ for $v \in K$ is a quadratic form on K . Choose the following coordinates

$$\begin{aligned} L &= \{(x, y) | x, y \in \mathbb{R}^n\}, K = \{(F_1(z), z) | z \in \mathbb{C}^n\} \text{ and} \\ L' &= \{(\overline{w}, F_1^-(\overline{w})) | w \in \mathbb{C}^n\} = \{(\overline{w}, \overline{F_1^{-1}(w)})\} = \{(\overline{F_1(w)}, \overline{w})\}, \end{aligned}$$

where we note $F_1^- = c \circ F_1^{-1} \circ c$, then it's easy to check that

$$f : K \rightarrow L : z \mapsto (x, y) = -(F_1(z) + \overline{F_1(z)}, z + \overline{z})$$

and for $v = (F_1(z), z)$

$$\begin{aligned} Q'(v) &= -\omega_0(F_1(z), \overline{F_1(z)}) + \omega_0(z, \overline{z}) \\ &= -\omega(F_1(z), F_1 \circ F_2^{-1}(\overline{z})) + \omega_0(z, \overline{z}) \\ &= \omega_0((1 - F_2)(z), \overline{z}) \\ &= Q(z). \end{aligned}$$

Together with (2.5), we are done.

We now show the existence of a trivialization Φ_z with $\Phi_{\overline{z}} = c_z \circ \Phi_z \circ dc$. Let V^\pm be the ± 1 eigen-bundle of dc action on $v|_{[-1, 1]}^* TM$, then they are transversal Lagrangian subbundles. Since $[-1, 1]$ is contractible, we trivialize V^+ and choose a section $\{e_r^j\}_{j=1}^n$ for $r \in [-1, 1]$ of the frame bundle. The induced trivialization of V^- is then given by $\{f_r^j\}_{j=1}^n$ where $\omega(e_r^j, f_r^k) = \delta_{kj}$. Then the trivialization Φ_r can be defined by $\{e_r^j, f_r^k\} \mapsto$ standard basis of $\mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n$. Then the trivialization Φ_r satisfies $\Phi_r = c_z \circ \Phi_r \circ dc$. Extend it to D_+^2 to obtain trivialization Φ_z for $z \in D_+^2$. Now define Φ_z for $z \in D_-^2$ by $\Phi_z = c_z \circ \Phi_{\overline{z}} \circ dc$ and $\Phi_{z \in D^2}$ gives a continuous trivialization of $v^* TM$ with the desired property. \square

2.4. Diagonal. For $(M, L) = (X \times X, \Delta)$, the doubling construction applies. Let $p_i : M \rightarrow X$, for $i = 1, 2$, be the projection to the i -th factor, then we obtain the following maps

$$\delta_i = p_i \circ \delta : \text{Map}(\Sigma, \partial\Sigma; M, \Delta) \rightarrow \text{Map}(\Sigma \cup_{\partial} \overline{\Sigma}; X)$$

which are natural isomorphism between the spaces of continuous maps. As special cases, the doubling gives isomorphisms of the path / loop spaces and the respective covering spaces:

$$\delta_i : \mathcal{P}_\Delta(M) \rightarrow \Omega^{(2)}(X) \text{ and } \delta_i : \widetilde{\mathcal{P}}_\Delta(M) \rightarrow \widetilde{\Omega}^{(2)}(X)$$

More explicitly, for example, for $l \in \mathcal{P}_\Delta(M)$ we write $l(t) = (l_1(t), l_2(t))$ then

$$(\delta_1(l))(t) = \begin{cases} l_1(t) & \text{for } t \in [0, 1] \\ l_2(2-t) & \text{for } t \in [1, 2] \end{cases}$$

This isomorphism extends to their corresponding normed completions as well. They also induce the isomorphisms $\delta_i : \pi_2(M, \Delta) \rightarrow \pi_2(X)$. The exact sequence of homotopy groups gives

$$\dots \rightarrow \pi_2(\Delta) \rightarrow \pi_2(M) \cong \pi_2(X) \times \pi_2(X) \xrightarrow{j} \pi_2(M, \Delta) \rightarrow \dots$$

Then we have for $\beta \in \pi_2(X)$:

$$\delta_1 \circ j(\beta, 0) = \delta_2 \circ j(0, -\beta) = \beta$$

It's straight forward to see that for $\beta \in \pi_2(X)$, $\delta_2 \circ \delta_1^{-1}(\beta) = -\beta = \tau(\beta)$. The isomorphism of homotopy group gives rise the isomorphism $\delta_i : \Gamma_\Delta \cong \Gamma_\omega$ as well as the corresponding Novikov rings. More precisely, for $a_\beta e^\beta \in \Lambda_\Delta$, we have

$$\delta_1(a_\beta e^\beta) = a_\beta e^{\delta_1(\beta)} \in \Lambda_\omega \text{ and } \delta_2(a_\beta e^\beta) = (-1)^{\frac{1}{2}\mu_\Delta(\beta)} a_\beta e^{\delta_2(\beta)} \in \Lambda_{-\omega}$$

then $\delta_2 \circ \delta_1^{-1} : \Lambda_\omega \rightarrow \Lambda_{-\omega}$ coincides with the isomorphism induced by reversing the symplectic structure on (X, ω) (cf. [2] §4).

Let $\{H_t, J_t\}_{t \in [0, 2]}$ be a pair of periodic Hamiltonian functions and compatible almost complex structures on (X, ω) , then

$$(\mathbb{H}_t, \mathbb{J}_t) = (H_t \oplus H_{2-t}, \mathbf{J}_t \oplus -\mathbf{J}_{2-t})$$

is a c -symmetric pair on $M = X \times X$, with symplectic form $\Omega = \omega \oplus (-\omega)$. Let $\{\phi_t\}_{t \in [0, 2]}$ denote the Hamiltonian isotopy generated by H_t on X , then $\{\psi_t = (\phi_t, \phi_{2-t} \circ \phi_2^{-1})\}_{t \in [0, 2]}$ is the Hamiltonian isotopy generated by \mathbb{H}_t on M . It follows that $x \in X$ is a non-degenerate fixed point of ϕ_2 iff $(x, x) \in \Delta$ is a non-degenerate fixed point of ψ_2 .

Let $(\mathbb{H}^1, \mathbb{J}^1)$ and $(\mathbb{H}^2, \mathbb{J}^2)$ be the two halves of (\mathbb{H}, \mathbb{J}) , i.e.

$$(\mathbb{H}^1, \mathbb{J}^1) = (\mathbb{H}_t, \mathbb{J}_t)_{t \in [0, 1]} \text{ and } (\mathbb{H}^2, \mathbb{J}^2) = (\mathbb{H}_{t+1}, \mathbb{J}_{t+1})_{t \in [0, 1]}$$

Let $\tilde{l} \in \tilde{\mathcal{P}}_\Delta M$ be a critical point of $a_{\mathbb{H}^1}$, then Lemma 2.3 implies that $\tilde{\gamma} = \delta(\tilde{l}) \in \tilde{\Omega}^{(2)}(M)$ is a critical point of $a_{\mathbb{H}}$. Let $\tilde{\gamma}_1 = p_1(\tilde{\gamma}) = \delta_1(\tilde{l}) \in \tilde{\Omega}(X)$, then it is a critical point of a_H . Similarly, $\tilde{\gamma}_2 = p_2(\tilde{\gamma})$ is a critical point of $a_{\underline{H}}$, with $\underline{H}_t = H_{2-t}$. Furthermore, the non-degeneracy of any one of these critical points implies that all the rest are also non-degenerate.

Lemma 2.6. *Suppose that all critical points involved are non-degenerate, then $\mu_{\mathbb{H}}(\tilde{\gamma}) = 2\mu_{\mathbb{H}}(\tilde{l})$. It follows that*

$$\mu_{\mathbb{H}}(\tilde{l}) = \mu_H(\tilde{\gamma}_1)$$

Proof: The critical point $\tilde{\gamma}$ is determined by its projection to the two factors, $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$. Notice that $\psi_t = (\phi_t, \phi_{2-t} \circ \phi_2^{-1})$, in (2.2), the identification Φ may be chosen such that it respects the decomposition $TM = p_1^*TX \oplus p_2^*TX$. Then it's clear that

$$\mu_{\mathbb{H}}(\tilde{\gamma}) = \mu_H(\tilde{\gamma}_1) + \mu_{\underline{H}}(\tilde{\gamma}_2)$$

Similar to Lemma 5.2 of [2], straight forward computation shows that

$$\mu_H(\tilde{\gamma}_1) = \mu_{\underline{H}}(\tilde{\gamma}_2) \Rightarrow \mu_{\mathbb{H}}(\tilde{\gamma}) = 2\mu_H(\tilde{\gamma}_1)$$

Now we only have to see that $\mu_{\mathbb{H}}(\tilde{\gamma}) = 2\mu_{\mathbb{H}}(\tilde{l})$. By Lemma 2.4 and Proposition 2.5, we only need to compute $\text{sign}(Q)$. Let $\gamma(0) = (x, x) \in \Delta$ and $\xi_1, \xi_2 \in T_x X$, then $\xi = (\xi_1, \xi_2) \in T_{\gamma(0)}M$ and

$$\begin{aligned} Q(\xi, \xi) &= \Omega((1 - d\psi_2)(\xi_1, \xi_2), (\xi_2, \xi_1)) \\ &= \omega((1 - d\phi_2)\xi_1, \xi_2) - \omega((1 - d\phi_2)\xi_2, \xi_1) \\ &= 2\omega((1 - d\phi_2)\xi_1, \xi_2) \end{aligned}$$

It follows that $\text{sign}(Q) = 0$. □

2.5. Proof of the lemma. The lemma follows from the following proposition and Proposition 4.2 of [2] which relates the quantum homology of opposite symplectic structures.

Proposition 2.7. δ_1 induces a natural isomorphism of the Floer theories

$$\delta_1 : FH_*(M, \Delta; \Omega) \simeq FH_*(X, \omega).$$

Proof: Using the notations from the last subsection, we first compare the action functionals. Let $\tilde{l} = [l, w] \in \tilde{\mathcal{P}}_\Delta M$ and $\tilde{\gamma}_1 = [\gamma_1, v_1]$ so that $\tilde{\gamma}_1 = \delta_1(\tilde{l})$, then

$$a_H([\gamma_1, v_1]) = - \int_{D^2} v_1^* \omega + \int_{[0,2]} H_t(\gamma_1(t)) dt = - \int_{D_+^2} w^* \Omega + \int_{[0,1]} \mathbb{H}_t(l(t)) dt = a_{\mathbb{H}}([l, w]).$$

Let $\{\xi_t\}_{t \in [0,2]}$ be a vector field along γ_1 , then $\{\eta_t = (\xi_t, \xi_{2-t})\}_{t \in [0,1]}$ is a vector field along l with $\eta_{0,1} \in T\Delta$ and vice versa. This gives the isomorphism on the tangent spaces:

$$D\delta_1 : T_l \tilde{\mathcal{P}}_\Delta M \rightarrow T_{\gamma_1} \Omega^{(2)}(X) : \eta \mapsto \xi.$$

It then follows that for $\eta, \eta' \in T_l \tilde{\mathcal{P}}_\Delta M$ and the corresponding ξ 's:

$$(\xi, \xi')_{\mathbb{J}} = \int_{[0,2]} \omega(\xi_t, J_t(\xi'_t)) dt = \int_{[0,1]} \omega(\xi_t, J_t(\xi'_t)) dt + \int_{[1,2]} \omega(\xi_{2-t}, J_{2-t}(\xi'_{2-t})) dt = (\eta, \eta')_{\mathbb{J}}.$$

From these we see that the Floer equations for the two theories are identified by δ_1 and the moduli spaces of smooth solutions are isomorphic for the two theories.

By Lemma 2.6, the gradings of the two theories coincide via δ_1 . We consider the orientations. Let's first orient the moduli spaces of holomorphic discs in (M, Δ) . Here we may assume that the almost complex structures involved are generic. The map δ_1 induces

$$\delta_1 : H_*(M, \Delta) \rightarrow H_*(X)$$

as well as the maps between the moduli spaces of (parametrized) holomorphic objects (discs or spheres):

$$\delta_1 : \widetilde{\mathcal{M}}(M, \Delta; \mathbb{J}, B) \rightarrow \widetilde{\mathcal{M}}(X; J, \delta_1(B)).$$

The map δ_1 is an isomorphism. We then put the induced orientation on the moduli space of discs. The moduli spaces of caps are similarly related by δ_1 and the orientations for a preferred basis on either theory can be chosen to be compatible with respect to δ_1 . It then follows that the orientations of the theories coincide under δ_1 .

To identify the two theories in full, we study the compactifications of the moduli spaces, in particular the compactifications by bubbling off holomorphic discs/spheres. The partial compactification given by the broken trajectories is naturally identified by δ_1 and the identification of the Floer equations.

Consider next the moduli spaces of holomorphic discs in (M, Δ) . The map δ_1 defined for the moduli spaces above extends to objects with marked points, which, for spheres, are along $\mathbb{RP}^1 \subset \mathbb{CP}^1$ while for the discs, are along the boundary:

$$\delta_1 : \widetilde{\mathcal{M}}_k(M, \Delta; \mathbb{J}_i, B) \rightarrow \widetilde{\mathcal{M}}_k(X; J_i, \delta_1(B)) \text{ for } i = 0, 1.$$

When we pass to the unparametrized moduli spaces, we also denote the induced map δ_1 . Next, we consider the evaluation maps from the moduli spaces of objects with 1-marked point:

$$ev^\Delta : \mathcal{M}_1(M, \Delta; \mathbb{J}_i, B) \rightarrow \Delta \text{ and } ev : \mathcal{M}_1(X; J_i, \delta_1(B)) \rightarrow X.$$

Let $p_1 : \Delta \rightarrow X$ be the natural projection, then we see that

$$p_1 \circ ev^\Delta = ev \circ \delta_1.$$

In particular, the image of the evaluation map ev^Δ has at most the same dimension as that of ev (in fact, they coincide via p_1):

$$\dim_{\mathbb{R}} = 2c_1(TX)(B) + 2n - 4.$$

The bubbling off of spheres are similar. The Floer theory $FH_*(X, \omega)$ is well defined and it follows that $FH_*(M, \Delta; \Omega)$ is well defined as well and they are isomorphic. \square

Recall from [2] (Proposition 5.5) that the Lagrangian Floer theories of (M, Δ, Ω) and $(M, \Delta, -\omega)$ are related by an isomorphism

$$\tau_* : FH_*(M, \Delta, \Omega; \mathbb{H}, \mathbb{J}) \rightarrow FH_*(M, \Delta; -\Omega; \mathbb{H}, \mathbb{J})$$

where $\mathbb{H}_t = \mathbb{H}_{2-t}$ and $\mathbb{J}_t = -\mathbb{J}_{2-t}$ here. We observe that the involution c on M identifies the tuples:

$$c : (M, \Delta, -\Omega; \mathbb{H}, \mathbb{J}) \rightarrow (M, \Delta; \Omega; \mathbb{H}, \mathbb{J})$$

and the induced map of c on Floer homology composed with τ_* is the identity map. Now Lemma 1.1 is given by the following diagram

$$\begin{array}{ccccc} FH_*(M, \Delta; \Omega) & \xrightarrow{\tau_*} & FH_*(M, \Delta; -\Omega) & \xrightarrow{c_*} & FH_*(M, \Delta; \Omega) \\ \delta_1 \downarrow \vee & & \delta_1 \downarrow \vee & & \delta_2 \downarrow \vee \\ FH_*(X, \omega) & \xrightarrow{\tau} & FH_*(X, -\omega) & & \end{array}$$

The commutativity of the left square follows from the discussion of reversing the symplectic structure in [2] (§4 – 5), while it's obvious that the right triangle commutes.

Corollary 2.8. *The half pair of pants product is well defined for $FH_*(M, \Delta)$ and it has a unit.*

Proof: Everything is induced from $FH_*(X, \omega)$ using the map δ_1 . \square

3. SEIDEL ELEMENTS AND THE ALBERS MAP

Let $\Omega_0\text{Ham}(M, \Omega)$ be the space of loops in $\text{Ham}(M, \Omega)$ based at 1. It's a group under pointwise composition. In $\Omega_0\text{Ham}(M, \Omega)$, a loop g is *split* if $g = (g_1, g_2)$ is in the image of the natural maps

$$\Omega_0\text{Ham}(X, \omega) \times \Omega_0\text{Ham}(X, -\omega) \rightarrow \Omega_0\text{Ham}(M, \Omega)$$

Otherwise, it is *non-split*. Similarly, such notions are defined for the π_1 of the Hamiltonian groups.

3.1. Split loops. In Seidel [6], the covering space $\tilde{\Omega}_0\text{Ham}(M, \Omega)$ is defined as

$$\tilde{\Omega}_0\text{Ham}(M, \Omega) := \left\{ (g, \tilde{g}) \in \Omega_0\text{Ham}(M, \Omega) \times \text{Homeo}(\tilde{\Omega}(M)) \mid \tilde{g} \text{ lifts the action of } g \right\}$$

with covering group Γ_Ω . We use \tilde{g} to denote an element in $\tilde{\Omega}_0\text{Ham}(M, \Omega)$. The results in [2] imply that, similar to [6], \tilde{g} defines a homomorphism $FH_*(\tilde{g})$ of $FH_*(M, \Delta)$ as a module over itself. Recall that $\delta_1 : \Gamma_\Delta \cong \Gamma_\omega$. Moreover, in the homotopy exact sequence

$$\dots \rightarrow \pi_2(\Delta) \xrightarrow{i} \pi_2(M) \rightarrow \pi_2(M, \Delta) \rightarrow \dots$$

we have $\text{img}(i) \subset \ker c_1 \cap \ker \Omega$, from which it follows that $\Gamma_\Omega \cong \Gamma_\Delta$.

In the following, we parametrize the loops in $\Omega_0\text{Ham}(X, \omega)$ by $[0, 2]$ and those in $\Omega_0\text{Ham}(M, \Omega)$ by $[0, 1]$. For $\alpha \in \Omega_0\text{Ham}(X, \omega)$, define the reparametrization $\alpha^{(\frac{1}{2})}(t) = \alpha(2t)$ for $t \in [0, 1]$. The natural injective map

$$i_+ : \Omega_0\text{Ham}(X, \omega) \rightarrow \Omega_0\text{Ham}(M, \Omega) : \alpha \mapsto \alpha_+ = (\alpha^{(\frac{1}{2})}, 1)$$

lifts to an injective map \tilde{i}_+ on the corresponding covering spaces (see the proof of Proposition 3.1). For $\tilde{\alpha} \in \tilde{\Omega}_0\text{Ham}(X, \omega)$, let $\tilde{\alpha}_+ = \tilde{i}_+(\tilde{\alpha}_+) \in \tilde{\Omega}_0\text{Ham}(M, \Omega)$ and $\tilde{\alpha}_- = \tilde{i}_-(\tilde{\alpha})$ where \tilde{i}_- is the lifting of

$$i_- : \Omega_0\text{Ham}(X, -\omega) \rightarrow \Omega_0\text{Ham}(M, \Omega) : \alpha \mapsto \alpha_- = (1, (\alpha^-)^{(\frac{1}{2})})$$

We note that $\tilde{\alpha}_\bullet$ is determined by the image of any element in $\tilde{\Omega}_0(M)$ by the unique lifting property of covering space. Take the trivial loop $p = (x, y) \in M$, then $x \in M$ is a trivial loop in $\Omega_0(X)$. Let $\tilde{\alpha}(\tilde{x}) = [\alpha(x), w] \in \tilde{\Omega}_0(X)$, where $\tilde{x} = [x, x] \in \tilde{\Omega}_0(X)$. Then $\tilde{\alpha}_+(\tilde{p}) = [(\alpha^{(\frac{1}{2})}(x), y), w \times \{y\}]$

Proposition 3.1. *The following diagram commutes*

$$\begin{array}{ccc}
 FH_*(M, \triangle; \Omega) & \xrightarrow{\delta_1} & FH_*(X, \omega) \\
 \downarrow FH_*(\tilde{\alpha}_{\pm}) & & \downarrow FH_*(\tilde{\alpha}) \\
 FH_*(M, \triangle; \Omega) & \xrightarrow{\delta_1} & FH_*(X, \omega)
 \end{array}$$

A similar diagram is commutative with δ_2 and $FH_*(\tilde{\alpha}^-)$ in places of δ_1 and $FH_*(\tilde{\alpha})$.

Proof: We describe the case for $\tilde{\alpha}_+$ and $\tilde{\alpha}_-$ is similar. Let $\tilde{l} \in \tilde{\mathcal{P}}_{\triangle} M$ and $\tilde{\gamma} = \delta_1(\tilde{l}) \in \tilde{\Omega}^{(2)}(X)$. By definition we have $l(t) = (\gamma(t), \gamma(2-t))$ for $t \in [0, 1]$ and h_1 acts on l by

$$(\alpha_+ \circ l)(t) = (\alpha_{2t} \circ \gamma(t), \gamma(2-t))$$

Then

$$(\delta_1(\alpha_+ \circ l))(t) = \begin{cases} \alpha_{2t}(\gamma(t)) & \text{for } t \in [0, 1] \\ \gamma(t) & \text{for } t \in [1, 2] \end{cases}$$

which implies that

$$\delta_1(\alpha_+ \circ l) = (\alpha^{(\frac{1}{2})} \# 1) \circ \gamma = (\alpha^{(\frac{1}{2})} \# 1) \circ \delta_1(l)$$

Notice that $\alpha^{(\frac{1}{2})} \# 1$ and α differ by a reparametrization. The equality above lifts to the covering of the loop spaces and gives a chain level identity for the respective Floer theories. In particular

$$\delta_1 \circ FH_*(\tilde{\alpha}_+) = FH_*(\tilde{\alpha}) \circ \delta_1$$

□

For $\alpha \in \Omega_0 \text{Ham}(X, \omega)$, let $\tilde{\alpha}$ be a lifting to $\tilde{\Omega}_0 \text{Ham}(X, \omega)$. The corresponding Seidel element is

$$\Psi^X(\tilde{\alpha}) := FH_*(\tilde{\alpha})(1) \in FH_*(X, \omega)$$

where 1 is the unit of the pair of pants product. Moreover, for any other lifting $\tilde{\alpha}'$ of α , there is $B \in \Gamma_{\omega}$ such that

$$\Psi^X(\tilde{\alpha}') = e^B \Psi^X(\tilde{\alpha})$$

Similarly, the Lagrangian Seidel element is given by

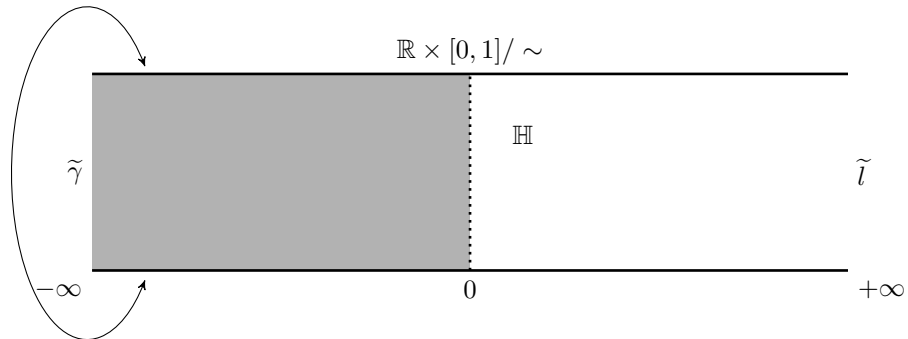
$$\Psi^{\triangle}(\tilde{\alpha}_+) = \Psi^{\triangle}(\tilde{\alpha}_-) = FH_*(\tilde{\alpha}_+)(1)$$

where 1 is the unit of the half pair of pants product.

Corollary 3.2. *For $\tilde{\alpha}$, $\tilde{\alpha}_{\pm}$ as given above we have $\delta_1(\Psi^{\triangle}(\tilde{\alpha}_+)) = \delta_1(\Psi^{\triangle}(\tilde{\alpha}_-)) = \Psi^X(\tilde{\alpha})$.* □

Since any split loop is the product of $(\alpha, 1)$ and $(1, \alpha')$, it follows that the split loops in $\Omega_0 \text{Ham}(M, \Omega)$ generate Seidel elements in $FH_*(X, \omega)$.

3.2. The Albers' map. Here we argue that the Albers' map is well defined for the example under consideration, where $X = (S^2 \times S^2, \omega_0 \oplus \lambda \omega_0)$ with $\lambda \in (1, 2]$. Recall that the map $\mathcal{A} : FH_*(M, \Omega) \rightarrow FH_*(M, \triangle; \Omega)$ is defined by counting of maps from the “chimney domain” $\mathbb{R} \times [0, 1] / \sim$:



where $(s, 0) \sim (s, 1)$ when $s \leq 0$, and the conformal structure at $(0, 0)$ is given by \sqrt{z} . In the figure above, the shaded left half of the strip has its two boundaries glued together forming a half infinite cylinder. At $-\infty$ it converges to $\tilde{\gamma}$, a critical point for the Floer theory $FH_*(M, \Omega)$, while at $+\infty$ it converges to \tilde{l} , a critical point for the Floer theory $FH_*(M, \Delta; \Omega)$.

In [1], the map \mathcal{A} is defined for monotone Lagrangians. Here, (M, Δ) is not monotone because

$$c_1(TM)((01\overline{00}) - (10\overline{00})) = 0 \text{ while } \omega((01\overline{00}) - (10\overline{00})) = \lambda - 1 > 0$$

On the other hand, for generic ω -compatible J on X , the class $(01\overline{00}) - (10\overline{00})$ is not represented by J -holomorphic spheres. In fact, the space of non-generic J 's has codimension 2. We choose such a generic pair (\mathbf{H}, \mathbf{J}) (for the Floer theory $FH_*(X, \omega)$) then the corresponding c -symmetric pair (\mathbb{H}, \mathbb{J}) on (M, Ω) is also generic for the Floer theories $FH_*(M, \Omega)$ and $FH_*(M, \Delta; \Omega)$. Since there is no holomorphic disc with non-positive Maslov number, the compactification of the 0-dimensional ‘‘chimney’’ moduli spaces would not contain disc bubblings. Similarly, we see that sphere bubblings can also be ruled out. It then follows that the map \mathcal{A} is well-defined.

3.3. Non-split loops. We showed that in $\Omega_0\text{Ham}(M, \Omega)$, there could be non-split loops, by computing directly the corresponding Seidel elements in $QH_*(M, \Omega)$. For such loops, Proposition 3.1 does not apply. On the other hand, let $g \in \Omega_0\text{Ham}(M, \Omega)$ be a non-split loop and \tilde{g} be a lifting to $\tilde{\Omega}_0\text{Ham}(M, \Omega)$, then it defines a Seidel element $\Psi^M(\tilde{g}) \in FH_*(M, \Omega)$. The Albers’ map [1] \mathcal{A} relates $FH_*(M, \Omega)$ and $FH_*(M, \Delta)$ when it’s well defined, in which case, we have

$$\mathcal{A} \circ \Psi^M(\tilde{g}) = \Psi^\Delta(\tilde{g}) \in FH_*(M, \Delta) \text{ and } \delta_1 \circ \mathcal{A} \circ \Psi^M(\tilde{g}) \in FH_*(X, \omega)$$

Consider $(X, \omega) = (S^2 \times S^2, \omega_0 \oplus \lambda\omega_0)$ for $\lambda \in (1, 2]$ and compute $\delta_1 \circ \mathcal{A} \circ \Psi^M(\tilde{g})$ for a non-split loop g . Also recall that McDuff [4] showed that liftings of the loops in the Hamiltonian group may be chosen such that

$$\Psi^X : \pi_1\text{Ham}(X, \omega) \rightarrow QH_*(X, \omega) : \alpha \mapsto \tilde{\alpha} \mapsto \Psi_\alpha^X := \Psi^X(\tilde{\alpha})$$

is a group homomorphism. Let $\psi = \Psi_{S'}^M$, i.e.

$$\psi = [(01\overline{11}) - (11\overline{10})] e^{\frac{1}{2}(1000) + h[(0001) + (1000)]}$$

To compute $\delta_1 \circ \mathcal{A}(\psi)$, we note first that $\delta_1 \circ \mathcal{A}$ is linear with respect to the identifications of the Novikov rings. Consider the following Seidel elements of split loops:

$$\Psi_{R_1}^M = (01\overline{11})e^{\frac{1}{2}(1000)} \text{ and } \Psi_{R_2}^M = -(11\overline{10})e^{-\frac{1}{2}(0001)}$$

then

$$\delta_1 \circ \mathcal{A} \circ \Psi_{R_1}^M = \delta_1(\Psi^\Delta(R_1)) = \Psi^X(r_1) = (01)e^{\frac{1}{2}(10)}$$

$$\delta_1 \circ \mathcal{A} \circ \Psi_{R_2}^M = \delta_1(\Psi^\Delta(\overline{R}_2)) = \Psi^X(r_2) = (10)e^{\frac{1}{2}(01)}$$

where we use r_i to denote the rotation of the i -th S^2 factor of X . In particular, we recall that via the identifications of Novikov rings,

$$e^{\frac{1}{2}(1000)} \mapsto e^{\frac{1}{2}(10)} \text{ and } e^{-\frac{1}{2}(0001)} \mapsto e^{\frac{1}{2}(10)}$$

It follows that

$$\delta_1 \circ \mathcal{A}(\psi) = [(01) + (10)] e^{\frac{1}{2}(10) + h[(10) - (01)]} = \Psi^X(s)$$

where s represents the element of infinite order in $\pi_1\text{Ham}(X, \omega)$. In summary, we showed

Proposition 3.3. *Under $\delta_1 \circ \mathcal{A}$ the Seidel elements of the non-split loops in $\text{Ham}(M, \Omega)$ constructed in [3] map to the Seidel elements of the loops of infinite order in $\text{Ham}(X, \omega)$. \square*

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